Congruent Reductions of Bilinear Forms.

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The following paper contains an account and a slight extension of a method due to Kronecker,* which, in the first place, was employed for the reduction of two quadratic forms; this method seems to have been used by no other writer, although in some ways it is the simplest that has been proposed. Here I have applied the method to four cases of reductions: (i) two symmetric forms (the same as Kronecker's case of two quadratics); (ii) a symmetric and an alternate form; (iii) two alternate forms, and (iv) two Hermite's forms. In cases (i)–(iii) the substitutions are congruent, while in (iv) they are conjugate imaginaries.

In §§1, 2, I have explained a method† for the reduction of a single form (alternate or symmetric); in §3, I have explained Kronecker's procedure for reducing two quadratic forms, using the results obtained in §§1, 2; the method is here put into such a shape that it can be applied to cases (ii), (iii) as well as (i). This way of explaining the reduction of two quadratics is suggested by Kronecker in an addition ("Nachtrag," Ges. Werke, Bd. 1, p. 397) to the first paper quoted. I have added a supplementary method to fill up a gap which presents itself in applying Kronecker's method to case (ii). In §4 is given a list of the reduced forms obtained for cases (i)–(iii). In §5, I have considered case (iv) somewhat briefly, as it can be obtained from the other three cases.

It may be convenient to indicate here the principal papers dealing with the problems to be considered.

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* Berliner Monatsberichte, 1874, p. 59 = Ges. Werke, Bd. 1, p. 349.

† This method for symmetric forms is the same as Kronecker's in the paper just quoted; the method for alternate forms is essentially the same as Kronecker's in a second paper (Berichte, p. 397 = Ges. Werke, p. 421).

Case (i). Weierstrass, Berliner Monatsberichte, 1858, p. 207, and 1868, p. 310 = Ges. Werke, Bd. 1, p. 233 and Bd. 2, p. 19.

Kronecker, Berliner Monatsberichte, 1868, p. 339 = Ges. Werke, Bd. 1. p. 163.

Berliner Sitzungsberichte, 1890, pp. 1225, 1375; 1891, pp. 9, 33, and the first papers quoted above.

Darboux, Liouville's Journal, t. 19 (sér. 2), 1874, p. 347.

Jordan, ibid., p. 397.

Case (ii). Kronecker, in the second paper* quoted above (i. e., Ges. Werke, Bd. 1, p. 421).

Frobenius, Berliner Sitzungsberichte, 1896, p. 7.

Case (iii). Frobenius, Crelle's Journal, Bd. 86, 1879, p. 140 (§§7, 13), and in the paper last quoted.

E. v. Weber, Münchener Sitzungsberichte, 1898, p. 369.

Case (iv). Alf. Loewy, Crelle's Journal, Bd. 122, 1900, p. 53.

Frobenius's paper (Berl. Ber., 1896, p. 7) contains a general theorem that if any two substitutions are known to change two forms A, B into two others C, D, then a congruent substitution can be deduced, which will make the same transformation, provided that A, C are both symmetric or both alternate, and that B, D have the same property. Thus (by virtue of Weierstrass's general theory), if A, B are given, they can be transformed into C, D by a *congruent* substitution, provided that the invariant-factors of $|\lambda A - B|$, $|\lambda C - D|$ are the same; this condition is obviously *necessary*, but Frobenius proves that it is also *sufficient*. In a paper recently published,† I have shown how to modify Weierstrass's process so as to obtain the congruent substitutions directly, thus giving an independent verification of Frobenius's results.

Apart from the immediate algebraical interest of these problems, they have certain applications, some of which I have indicated elsewhere,‡ and I hope that the solution given here may not be found superfluous.

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* Kronecker's object in this paper was to reduce a single bilinear form by congruent substitutions; not a symmetric and an alternate form simultaneously. The two problems are, however, essentially the same. The origin of the problem was connected with Weierstrass's generalized theta-functions (cf. Kronecker, Berliner Monatsberichte, 1866, p. 597 = Crelle, Bd. 68. p. 273 = Werke. Bd. 1, p. 143).

† Proc. Lond. Math. Soc., vol. XXXII, 1900, p. 321. This paper also contains a solution of Case (iv) and some applications to automorphic substitutions of bilinear forms.

‡ See last reference; two papers on dynamical applications will appear shortly in the same Proceedings.

1. Let $A = \Sigma a_{rs} x_r y_s$ be a bilinear function of the $2n$ variables $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ and now examine the form

$$B = \begin{vmatrix} a_{11}, & \ldots, & a_{1k}, & \dfrac{\partial A}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & \ldots, & a_{kk}, & \dfrac{\partial A}{\partial x_k} \\ \dfrac{\partial A}{\partial y_1}, & \ldots, & \dfrac{\partial A}{\partial y_k}, & A \end{vmatrix}.$$

Differentiating with respect to x_r, we see that (since x_r appears only in the last row of B),

$$\frac{\partial B}{\partial x_r} = \begin{vmatrix} a_{11}, & \ldots, & a_{1k}, & \dfrac{\partial A}{\partial x_1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & \ldots, & a_{kk}, & \dfrac{\partial A}{\partial x_k} \\ a_{r1}, & \ldots, & a_{rk}, & \dfrac{\partial A}{\partial x_r} \end{vmatrix},$$

which vanishes identically if $r = 1, 2, \ldots, k$. Thus B does not contain $x_1, x_2, \ldots, x_k$; and by a similar method B does not contain $y_1, y_2, \ldots, y_k$. In particular, if $k = n$, we have $B = 0$, and we obtain a familiar form for A.

Again, by examining $\dfrac{\partial^2 B}{\partial x_r \partial y_s}$, it is easy to prove that B vanishes identically if *all* the $(k+1)$-rowed determinants of the type

$$\begin{vmatrix} a_{11}, & \ldots, & a_{1k}, & a_{1s} \\ \cdots & \cdots & \cdots & \cdots \\ a_{k1}, & \ldots, & a_{kk}, & a_{ks} \\ a_{r1}, & \ldots, & a_{rk}, & a_{rs} \end{vmatrix} \qquad (r, s > k)$$

are zero; and, in this case, if the k-rowed determinant

$$\begin{vmatrix} a_{11}, & \ldots, & a_{1k} \\ \cdots & \cdots & \cdots \\ a_{k1}, & \ldots, & a_{kk} \end{vmatrix}$$

is not zero, A is expressible in terms of the $2k$ quantities

$$\frac{\partial A}{\partial x_1}, \ldots, \frac{\partial A}{\partial x_k}, \frac{\partial A}{\partial y_1}, \ldots, \frac{\partial A}{\partial y_k}.$$

In particular, if the determinant $|A|$ of the form A is zero, it is possible to express A in terms of $2(n-1)$ variables, or fewer variables, in case *all* the first minors are zero too.

We shall have occasion to apply the above theorem in various forms. Take in the first place $k=1$, and then

$$A=\frac{1}{a_{11}}\frac{\partial A}{\partial y_1}\frac{\partial A}{\partial x_1}+\frac{1}{a_{11}^2}B,$$

where B does not contain x_1 or y_1.

Supposing that the variables are not subject to the condition of undergoing congruent substitutions, we can always apply this method, provided that x_1 does appear in A, for then x_1 multiplies at least one y, and we may call this y, y_1, and so $a_{11}\neq 0$, which is the condition for the application of our result. Pass then to the case of congruent substitutions; here A will be either symmetrical or alternate, and we proceed to examine these cases.

First, take a symmetrical form, then, if $a_{11}\neq 0$, we can write

$$A=\frac{1}{a_{11}}\frac{\partial A}{\partial y_1}\frac{\partial A}{\partial x_1}+\frac{1}{a_{11}^2}B,$$

where B is again symmetrical and does not contain x_1 or y_1.

But if $a_{11}=0$, we must proceed to the result given by $k=2$, and then we find (since $a_{12}=a_{21}$),

$$A=\frac{1}{a_{12}}\left(\frac{\partial A}{\partial y_1}\frac{\partial A}{\partial x_2}+\frac{\partial A}{\partial y_2}\frac{\partial A}{\partial x_1}\right)-\frac{a_{22}}{a_{12}^2}\frac{\partial A}{\partial y_1}\frac{\partial A}{\partial x_1}-\frac{1}{a_{12}^2}B.$$

Thus, if we write

$$X_2=\frac{\partial A}{\partial y_1},\qquad X_1=\frac{1}{a_{12}}\frac{\partial A}{\partial y_2}-\frac{1}{2}\frac{a_{22}}{a_{12}^2}\frac{\partial A}{\partial y_1},$$

$$Y_2=\frac{\partial A}{\partial x_1},\qquad Y_1=\frac{1}{a_{12}}\frac{\partial A}{\partial x_2}-\frac{1}{2}\frac{a_{22}}{a_{12}^2}\frac{\partial A}{\partial x_1},$$

we have

$$A=X_1Y_2+X_2Y_1-\frac{1}{a_{12}^2}B,$$

where B is a symmetrical form, not containing x_1, y_1, x_2, y_2. The condition that this method should be applicable is that $a_{12}\neq 0$, and when x_1, y_1 have been fixed upon, it is always possible to find x_2, y_2, so that $a_{12}\neq 0$, provided that x_1, y_1 do

actually appear in A. It will be observed that X_1, Y_1 may contain all the variables, while X_2, Y_2 do not contain x_1, y_1.

In the alternate case $a_{11}=0$, $a_{22}=0$, and $a_{12}=-a_{21}$; thus taking $k=2$, we find

$$A=\frac{1}{a_{12}}\left(\frac{\partial A}{\partial x_1}\frac{\partial A}{\partial y_2}-\frac{\partial A}{\partial x_2}\frac{\partial A}{\partial y_1}\right)+\frac{1}{a_{12}^2}B.$$

Now, if we write

$$X_1=\frac{1}{a_{12}}\frac{\partial A}{\partial y_2}, \qquad Y_1=-\frac{1}{a_{12}}\frac{\partial A}{\partial x_2},$$
$$X_2=\frac{\partial A}{\partial y_1}, \qquad Y_2=-\frac{\partial A}{\partial x_1},$$

the substitutions are congruent and the result is

$$A=-X_1Y_2+X_2Y_1+\frac{1}{a_{12}^2}B,$$

where B is an alternate form not containing x_1, y_1, x_2, y_2; here X_1 contains x_1 and may contain all the other x's except x_2, while X_2 contains x_2 and may contain all but x_1.

Suppose, now, that the variables are divided in any way into two sets G, H; owing to the congruent conditions, it will be necessary to suppose that y_r belongs to the same set as x_r. Let us now impose the condition that the variables in H can only be linearly combined amongst themselves, and that no variables from G may be added to the variables in H; on the other hand, variables from H may be added freely to those in G. Then, applying the methods just explained,* we get a series of reduced terms (in which each variable occurs once only) of the type $(x_ry_s+x_sy_r)$ or $(x_ry_s-x_sy_r)$, according as the form considered is symmetric or alternate; in the process, the sets of variables G, H will be further subdivided into G_1, G_2, H_1, H_2. The characteristic of G_1 is that its variables multiply each other only, while those of G_2 multiply those of H_1; the variables in H_2 multiply each other. We may write the reduced form symbolically

$$A=(G_1)+(G_2H_1)+(H_2).$$

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* The pair of variables represented above by x_1, y_1 should be taken always from the variables in G, then X_1, Y_1 will replace x_1, y_1 in G; and if we have the reduced form $(X_1Y_2+X_2Y_1)$ or $(X_1Y_2-X_2Y_1)$, we have to examine X_2, Y_2, then if any variables from G appear in X_2, Y_2, the quantities X_2, Y_2 will be taken as new variables, replacing two of those in G. But it may happen that X_2, Y_2 contain only variables from H (as they do *not* contain x_1, y_1), and then they form part of H_1; in this case X_1, Y_1 are variables belonging to the substitution G_2.

2. Again, returning to our original theorem, let us take $k = n - 1$, and

$$B = \begin{vmatrix} a_{22}, & \ldots, & a_{2n}, & \dfrac{\partial A}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n2}, & \ldots, & a_{nn}, & \dfrac{\partial A}{\partial x_n} \\ \dfrac{\partial A}{\partial y_2}, & \ldots, & \dfrac{\partial A}{\partial y_n}, & A \end{vmatrix},$$

then B will contain only x_1 and y_1. To evaluate B, differentiate with respect to x_1 and we see that

$$\frac{\partial B}{\partial x_1} = y_1 D, \text{ or } B = D x_1 y_1,$$

where D is the determinant $|a_{rs}|$, $(r, s = 1, 2, \ldots, n)$. Thus we have

$$AD_{11} = Dx_1y_1 - \begin{vmatrix} a_{22}, & \ldots, & a_{2n}, & \dfrac{\partial A}{\partial x_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n2}, & \ldots, & a_{nn}, & \dfrac{\partial A}{\partial x_n} \\ \dfrac{\partial A}{\partial y_2}, & \ldots, & \dfrac{\partial A}{\partial y_n}, & 0 \end{vmatrix},$$

where D_{11} is the minor of a_{11} in D.

But, if D_{11} is not zero, it is possible to find $X_2, \ldots, X_n$ so as to satisfy

$$a_{22}X_2 + \ldots + a_{n2}X_n = \frac{\partial A}{\partial y_2},$$
$$\cdots\cdots\cdots\cdots$$
$$a_{2n}X_2 + \ldots + a_{nn}X_n = \frac{\partial A}{\partial y_n},$$

and then each of the $(n-1)$ differences $(X_2 - x_2), \ldots, (X_n - x_n)$ is a multiple of x_1. Similarly, we can find $(n-1)$ y's to satisfy the equations

$$a_{22}Y_2 + \ldots + a_{2n}Y_n = \frac{\partial A}{\partial x_2},$$
$$\cdots\cdots\cdots\cdots$$
$$a_{n2}Y_2 + \ldots + a_{nn}Y_n = \frac{\partial A}{\partial x_n}.$$

Then we can clearly write

$$A = \frac{D}{D_{11}} x_1 y_1 + C,$$

where C is derived from A by writing zero for x_1, y_1 and X_r, Y_r for x_r, y_r $(r = 2, 3, \ldots, n)$. We see that the equations for X_r, Y_r may be put in the form $\frac{\partial C}{\partial Y_r} = \frac{\partial A}{\partial y_r}, \frac{\partial C}{\partial X_r} = \frac{\partial A}{\partial x_r}$. This method can, of course, be applied if $D = 0$, but cannot be applied if $D_{11} = 0$. As explained before, if there is no restriction as to congruent substitutions, we shall, in general, be able to arrange the y's corresponding to a given x, so that $D_{11} \neq 0$; an exceptional case may arise if *all* the first minors of D are zero; but then it is possible to reduce A so as to depend only on $(n - 2)$ pairs of variables (or fewer pairs).

But, in symmetrical or alternate cases, it cannot always be arranged, that $D_{11} \neq 0$, and we proceed to examine these cases. Suppose that A is symmetrical, then if $D_{11} \neq 0$, our result still holds and C is symmetrical; but if $D_{11} = 0$, while $D \neq 0$, consider the form (containing only x_1, y_1, x_2, y_2),

$$B = \begin{vmatrix} a_{33}, & \ldots, & a_{3n}, & \frac{\partial A}{\partial x_3} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n3}, & \ldots, & a_{nn}, & \frac{\partial A}{\partial x_n} \\ \frac{\partial A}{\partial y_3}, & \ldots, & \frac{\partial A}{\partial y_n}, & A \end{vmatrix},$$

We find on calculation that

$$\frac{\partial B}{\partial x_1} = y_1 D_{22} - y_2 D_{21}, \quad \frac{\partial B}{\partial x_2} = -y_1 D_{12} + y_2 D_{11},$$

where D_{12}, D_{21}, D_{22} are the minors of a_{12}, a_{21}, a_{22} respectively in D. But $D_{11} = 0$ and $D_{12} = D_{21}$ from symmetry and so

$$B = x_1 y_1 D_{22} - D_{12}(x_1 y_2 + x_2 y_1),$$

for B contains none of the variables $x_3, \ldots, x_n, y_3, \ldots, y_n$. When B is expanded in terms of A and the products $\frac{\partial A}{\partial x_r} \frac{\partial A}{\partial y_s}$, the coefficient of A is a second principal minor of D; and the complementary minor of D is

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix}.$$

Hence the minor multiplying A in the expression for B is equal to

$$\frac{1}{D}\begin{vmatrix} D_{11}, & D_{12} \\ D_{21}, & D_{22} \end{vmatrix} = -D_{12}^2/D; \text{ (since } D_{11}=0).$$

Thus

$$A = \frac{D}{D_{12}}(x_1y_2+x_2y_1) - \frac{DD_{22}}{D_{12}^2}x_1y_1 + \frac{D}{D_{12}^2}\begin{vmatrix} a_{33} & , \dots, & a_{3n}, & \frac{\partial A}{\partial x_3} \\ \dots & \dots & \dots & \dots \\ a_{n3} & , \dots, & a_{nn}, & \frac{\partial A}{\partial x_n} \\ \frac{\partial A}{\partial y_3}, & \dots, & \frac{\partial A}{\partial y_n}, & 0 \end{vmatrix},$$

or, if we write $X_2 = x_2 - \frac{1}{2}\frac{D_{22}}{D_{12}}x_1$, $Y_2 = y_2 - \frac{1}{2}\frac{D_{22}}{D_{12}}y_1$, the first terms in A can can be put in the shape

$$\frac{D}{D_{12}}(x_1Y_2 + y_1X_2).$$

Now, if $D_{12} \neq 0$, the second minor of D,

$$\begin{vmatrix} a_{33}, & \dots, & a_{3n} \\ \dots & \dots & \dots \\ a_{n3}, & \dots, & a_{nn} \end{vmatrix}$$

will not vanish, for this has been proved to be $-D_{12}^2/D$; and so the $(n-2)$ quantities $X_3, \dots, X_n$ can be found to satisfy the $(n-2)$ equations

$$a_{33}X_3 + \dots + a_{n3}X_n = \frac{\partial A}{\partial y_3},$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{3n}X_3 + \dots + a_{nn}X_n = \frac{\partial A}{\partial y_n},$$

and each of the differences $(X_3-x_3), \dots, (X_n-x_n)$ will be a linear function of x_1, x_2. By symmetry, we have similar equations to determine $Y_3, \dots, Y_n$, and then, on substitution in the expression for A, we find

$$A = \frac{D}{D_{12}}(x_1Y_2 + y_1X_2) + C,$$

where C is found from A by writing zero for x_1, y_1, x_2, y_2 and X_r, Y_r in place of x_r, y_r $(r=3, 4, \dots, n)$. Here again X_r, Y_r can be determined by the equations $\frac{\partial C}{\partial Y_r} = \frac{\partial A}{\partial y_r}$, $\frac{\partial C}{\partial X_r} = \frac{\partial A}{\partial x_r}$.

It will, of course, be possible always to find one minor D_{12} which does not vanish unless D is itself zero. But if D be zero, we shall be able to reduce A to depend only on $2(n-1)$ variables instead of $2n$.

Now, consider the alternate case; here if n be odd, the determinant D is zero; and if n be even, $D_{11}, D_{22}, \ldots$, which are skew-symmetrical determinants of odd order, vanish.

If n be odd, we can apply our first method without any special modification to show that A can be brought to a form containing only $2(n-1)$ variables. Thus we consider only the case when n is even. Suppose that $D \neq 0$ (or we could reduce the number of variables to $2(n-2)$), and then $D_{11}=0$, $D_{22}=0$, $D_{12}=-D_{21}$. So, for the form

$$B = \begin{vmatrix} a_{33}, & \ldots, & a_{3n}, & \dfrac{\partial A}{\partial x_3} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n3}, & \ldots, & a_{nn}, & \dfrac{\partial A}{\partial x_n} \\ \dfrac{\partial A}{\partial y_3}, & \ldots, & \dfrac{\partial A}{\partial y_n}, & A \end{vmatrix},$$

we find the value $B = (x_1 y_2 - x_2 y_1) D_{12}$. Just as before, we can find $X_3, \ldots, X_n$ to satisfy the $(n-2)$ equations

$$a_{33} X_3 + \ldots + a_{n3} X_n = \frac{\partial A}{\partial y_3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$a_{3n} X_3 + \ldots + a_{nn} X_n = \frac{\partial A}{\partial y_n},$$

and, owing to the alternate property ($a_{rs} = -a_{sr}$), it follows that $Y_3, \ldots, Y_n$ will satisfy

$$a_{33}\, Y_3 + \ldots + a_{3n}\, Y_n = \frac{\partial A}{\partial x_3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{n3}\, Y_3 + \ldots + a_{nn}\, Y_n = \frac{\partial A}{\partial x_n}.$$

Then we have, on substitution,

$$A = \frac{D}{D_{12}}(x_1 y_2 - x_2 y_1) + C,$$

where C is found by writing zero for x_1, y_1, x_2, y_2 in A, and X_r, Y_r for x_r, y_r $(r = 3, 4, \ldots, n)$.

We can express this result in a slightly different form by using Pfaffians; we know that D can be put in the shape

$$D = P^2,$$

where P is a rational function of the elements of the determinant; and the second principal minor, which is equal to $+D_{12}^2/D$, is also the square of a Pfaffian, say of P_1. Hence, $D_{12} = \pm PP_1$, and we can determine the sign of P_1 so as to give the upper sign, thus we shall have

$$A = \frac{P}{P_1}(x_1 y_2 - x_2 y_1) + C.$$

Let the variables be divided in any way into two sets H, K (as before, x_r, y_r belong to the same set); here suppose that the variables in H may only be combined with each other, while those in K may be combined with each other and with those in H as well. We apply the process given above: The sets then each subdivide into two; H into H_1, H_2; and K into K_1, K_2; the variables in H_1 (in the reduced forms) multiply other variables of the set H_1; those in H_2 and K_1 are multiplied together.* Thus using the same symbolical notation as before, we have

$$A = (H_1) + (H_2 K_1) + (K_2).$$

3.—*General account of Kronecker's Method of Reduction.*

Consider two forms A, B which may be either symmetric or alternate, so that each of them can be brought to a reduced form by means of congruent substitutions of the types already indicated. We shall suppose that A contains some variables that do not appear in B, while B also contains some that are not present in A. This divides the variables into three sets, G, H, K; G contains all the variables that appear in A and do not appear in B; K, those that are in B and not in A; H, those that are common to the two forms.

Applying the first method of reduction (§1) to A, the variables G are divided into G_1, G_2, and the variables H into $H_1 = L$ and $H_2 = M$. Of course,

*In the foregoing reduction, we select x_1, y_1 from the variables in H; if the reduced part is $(x_1 Y_2 \pm X_2 y_1)$, we examine X_2, Y_2; in case X_2, Y_2 contain only variables from H, we group x_1, y_1, X_2, Y_2 in H_1, but if X_2, Y_2 contain variables from K, x_1, y_1 will belong to H_2, and X_2, Y_2 to K_1.

in the reduction, the variables are modified, but in such a way that the new variables in H are linear functions only of the old variables in H. Then A takes the form $(G_1)+(G_2L)+(M)$, where, in the forms (G_1), (G_2L), each variable occurs once only.

Substitute these new variables in the form B and consider then the variables of B as belonging to the three sets L, M, K. We now apply the second method of reduction (§2) to B and we shall obtain a result of the form

$$(L_1)+(L_2K_1)+(L_3M_1)+(M_2)+(K_2),$$

the variables L dividing into L_1, L_2, L_3, M into M_1, M_2, and K into K_1, K_2. In this process the L's are modified only by other L's, and the M_1's only by M_1's and L's; further, in the terms (L_1), (L_2K_1), (L_3M_1), each variable occurs once only. Substitute the new variables in the expression for A; the additional terms introduced by the change of variables will either contain only M's or else will have some variable L as a factor; in the latter case the additional terms can be absorbed by modifying the variables G_2. Thus we may now write

$$\begin{aligned} A &= (G_1)+(G_2L)+(M) \\ B &= \qquad\qquad (L_1)+(L_2K_1)+(L_3M_1)+(M_2)+(K_2) \end{aligned}$$

in these we can pair off certain terms. Thus we remove

$$(G_1),\ (G_2L_1) \text{ and } (L_1),\ (G_2L_2) \text{ and } (L_2K_1),$$

and the remainders will take the forms

$$\begin{aligned} &(G_2L_3)+(M) \\ &(L_3M_1)+(M_2)+(K_2). \end{aligned}$$

Now, taking the terms (M) and $(M_2)+(K_2)$, we observe that they are of the same general type as A and B were at first, but contain fewer variables; and the groups of variables corresponding to the original G, H, K are here M_1, M_2, K_2. We can accordingly continue the process as given above, and in the continuation we alter none of the variables in those parts of A, B, which have been already reduced, except those which belong to the set M; those in M_1 may have to be replaced by linear functions of the M's, when we reduce the two (M) and $(M_2)+(K_2)$.

So far as the variables M_1 are altered by substitutions containing only themselves, we can make corresponding substitutions on the variables L_3 and G_2, so that

$$(G_2'L_3')=(G_2L_3) \text{ and } (L_3'M_1')=(L_3M_1).$$

Thus we have only to examine the effect of adding on variables from M_2 to the variables M_1; this will give, instead of

$$(L_3M_1)+(M_2)+(K_2),$$

the terms
$$(L_3M_1)+(L_3M_2)+(M_2)+(K_2).$$

Now, by using the second method of reduction, we can combine $(L_3M_2)+(M_2)$ so that, by adding on linear functions of the variables L_3 to the variables M_2, we get $(M_2')+(L_3)$, and then these can be combined with (L_3M_1) so that we may write

$$(L_3M_1)+(L_3M_2)+(M_2)=(L_3M_1')+(M_2'),$$

where now each M is modified by some of the variables L_3. Substituting in the remaining terms of A, we find that the additional terms so introduced have each one variable L_3 as a factor and so can be combined with the terms (G_2L_3) by introducing new variables G_2.

It follows that this method can be continued so long as there are variables in one form which do not appear in the other, and that when we have to stop. there can only be variables common to both forms in the parts which remain.

If the forms to be reduced are *both* symmetric or *both* alternate, the process of reduction, as just explained, can be applied to complete the whole reduction, For, if A, B are both symmetric (or alternate), then all forms of the family $(uA+vB)$ are symmetric (or alternate), and we can determine values of u, v for which the determinant $|uA+vB|$ vanishes; say $u_1:v_1$, $u_2:v_2$ are two values of the ratio $u:v$ for which the determinant is zero. Then u_1A+v_1B, u_2A+v_2B will be two forms, each a function of fewer variables than appear in the general form $uA+vB$. Thus u_1A+v_1B, u_2A+v_2B can be transformed so that each contains some variables that do not occur in the other, and the process already sketched can be completely carried out.

But if one of the forms (A, say) be symmetric while the other (B) is alternate, then the general form $uA+vB$ will be neither symmetric nor alternate, and thus the method of reduction described can be carried out only so long as $|A|=0$ or $|B|=0$. We shall now explain a process of reduction to cover the general case.*

* Kronecker describes his reduction of a single bilinear form by congruent substitutions as suggested by his method for two quadratics; but it differs considerably from that given here, although the fundamental ideas are the same.

By what has been proved, we can reduce a symmetric and an alternate form to the shapes

$$A = A_1 + A_2, \quad B = B_1 + B_2,$$

where A_1, B_1 are reduced forms and A_2, B_2, are forms of such a character that $|A_2| \neq 0$, $|B_2| \neq 0$; further A_2, B_2 will not contain the variables in A_1, B_1. Consequently, the number of pairs of variables in A_2, B_2 must be *even*.

Suppose then that we start from two forms A, B in $2m$ pairs of variables neither of whose determinants is zero; and let $\lambda = c$ be a root of the determinantal equation $|\lambda A + B| = 0$. Then, since $(cA + B)$ is a form with zero determinant, it depends on $(2m - 1)$ y's at most; and so we can choose our variables in such a way that one x (x_1 say) has no corresponding y present in $(cA + B)$, the substitutions being supposed congruent. Apply now to $(cA + B)$ the method of transformation explained in §1, for a single bilinear form; then

$$cA + B = 2cx_1' y_2' + \text{terms without } x_1',\ y_1' \text{ or } y_2',$$

where (since there is no term $x_1 y_1$ in $cA + B$), x_1' is a linear function of x's (containing x_1) and y_2' is a linear function of y's (not containing y_1). Interchange the x's and y's of the last equation, then, as this changes the sign of B,

$$cA - B = 2c\, x_2' y_1' + \text{terms without } x_1', x_2' \text{ or } y_1'.$$

Hence
$$A = x_1' y_2' + x_2' y_1' + \text{terms without } x_1', y_1' \text{ or the product } x_2' y_2',$$
$$B = c\,(x_1' y_2' - x_2' y_1') + \text{ terms without } x_1', y_1'.$$

Applying to A the method given in §1 for a single symmetrical form we can collect all the terms in x_2', y_2' together and write

$$A = x_1'' y_2' + x_2' y_1'' + \text{terms without } x_1'', y_1'', x_2', y_2',$$
$$B = c\,(x_1'' y_2' - x_2' y_1'') + \text{terms without } x_1'', y_1''.$$

Take next the pair of forms in $4(m - 1)$ variables obtained from A, B by putting $x_2' = 0$, $y_2' = 0$; we treat them in the same way. Proceeding thus we may write finally, dropping the accents,

$$A = (x_1 y_2 + x_2 y_1) + (x_3 y_4 + x_4 y_3) + \cdots + (x_{2m-1} y_{2m} + x_{2m} y_{2m-1}),$$
$$B - B_1 = c_1 (x_1 y_2 - x_2 y_1) + c_2 (x_3 y_4 - x_4 y_3) + \cdots + c_m (x_{2m-1} y_{2m} - x_{2m} y_{2m-1}),$$

where
$$B_1 = (x_2 \eta_2 - y_2 \xi_2) + \cdots (x_{2m-2} \eta_{2m-2} - y_{2m-2} \xi_{2m-2}),$$

and
ξ_r is a linear function of $x_{r+1}, x_{r+2}, \ldots, x_{2m}$,
η_r being the same function of $y_{r+1}, y_{r+2}, \ldots, y_{2m}$.

We now proceed to remove as many terms as possible from B_1.

Take the term $\alpha(x_2 y_3 - x_3 y_2)$,

and write

$$x_1 = x_1' - \mu x_3, \quad y_1 = y_1' - \mu y_3,$$
$$x_4 = x_4' + \mu x_2, \quad y_4 = y_4' + \mu y_2.$$

Then

$$x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3 = x_1' y_2 + x_2 y_1' + x_3 y_4' + x_4' y_3,$$

and

$$c_1(x_1 y_2 - x_2 y_1) + \alpha(x_2 y_3 - x_3 y_2) + c_2(x_3 y_4 - x_4 y_3)$$
$$= c_1(x_1' y_2 - x_2 y_1') + c_2(x_3 y_4' - x_4' y_3) + (\alpha + \mu c_1 - \mu c_2)(x_2 y_3 - x_3 y_2),$$

Or, if $\mu = \alpha / (c_2 - c_1)$, the term in question will be removed from B by the substitution proposed; but if $c_1 = c_2$, this will be no longer possible. We may accordingly regard A and B as reduced, except for terms of the type

$$(A) \qquad (x_1 y_2 + x_2 y_1) + \ldots + (x_{2p-1} y_{2p} + x_{2p} y_{2p-1})$$
$$(B) \qquad c(x_1 y_2 - x_2 y_1) + \ldots + c(x_{2p-1} y_{2p} - x_{2p} y_{2p-1}) + B_2.$$

where B_2 contains terms of the same type as B_1 before, but is limited to the variables $x_2, \ldots, x_{2p}, y_2, \ldots, y_{2p}$. Then consider a term in B_2 of the type

$$\alpha(x_2 y_{2q} - x_{2q} y_2),$$

and write

$$x_1 = x_1' + (\alpha/2c) x_{2q},$$
$$x_{2q-1} = x_{2q-1}' - (\alpha/2c) x_2,$$

with the corresponding substitution in the y's. Then the form of A is unaltered and the terms affected in B become

$$cy_2[x_1' + (\alpha/2c) x_{2q}] - cx_2[y_1' + (\alpha/2c) y_{2q}] + cy_{2q}[x_{2q-1}' - (\alpha/2c) x_2]$$
$$- cx_{2q}[y_{2q-1}' - (\alpha/2c) y_2] + \alpha(x_2 y_{2q} - x_{2q} y_2)$$
$$= c(x_1' y_2 - x_2 y_1') + c(x_{2q-1}' y_{2q} - x_{2q} y_{2q-1}').$$

By the same method we remove all terms with two even suffixes from B_2; so proceeding in this way, the only terms left in B_2 will be of the type

$$x_{2r} y_{2q+1} - y_{2r} x_{2q+1}, \qquad (q \overline{\gt} r).$$

Consider those in x_2, y_2; let us suppose that the first term of B_2 which contains x_2, y_2 is

$$x_2 y_3 - x_3 y_2;$$

this assumption may involve certain changes in the suffixes; but it will not

interchange the even and odd suffixes. It may also require a division and multiplication of certain variables by a constant. Consider next the pair of terms in B_2.

$$x_2 y_3 - x_3 y_2 + \alpha (x_2 y_{2q-1} - y_2 x_{2q-1}), \qquad (q > 2).$$

Write $$x_3' = x_3 + \alpha x_{2q-1}, \; x_{2q}' = x_{2q} - \alpha x_4,$$

then the form of A remains the same and these terms in B_2 become;

$$x_2 y_3' - x_3' y_2.$$

This substitution may add to the parts of B_2 which contain x_4, y_4 but will not otherwise alter B. Of course it may happen that B_2 contains no terms in x_2, y_2 and then we shall not have to reduce the terms $(x_1 y_2 + x_2 y_1)$, $c(x_1 y_2 - x_2 y_1)$ in B any further.

Continuing our process, we have finally A, B divided into groups of terms

$$\begin{array}{ll} (A) & (x_1 y_2 + x_2 y_1) + \ldots\ldots + (x_{2s-1} y_{2s} + x_{2s} y_{2s-1}), \\ (B) & c(x_1 y_2 - x_2 y_1) + \ldots\ldots + c(x_{2s-1} y_{2s} - x_{2s} y_{2s-1}), \\ & + (x_2 y_3 - x_3 y_2) + \ldots\ldots + (x_{2s-2} y_{2s-1} - x_{2s-1} y_{2s-2}). \end{array}$$

In particular for $s = 1$ we have

$$\begin{array}{ll} (A) & (x_1 y_2 + x_2 y_1) \\ (B) & c(x_1 y_2 - x_2 y_1). \end{array}$$

It is readily seen that these terms correspond to invariant-factors $(\lambda - c)^s$ $(\lambda + c)^s$, of $|\lambda A - B|$.

4.—*Lists of Reduced Forms.*

We now proceed to enumerate all the possible types of reduced forms at which we arrive by following Kronecker's process as just explained. There are three cases, each of which has to be examined separately; i. e., two symmetric forms one symmetric and one alternate form; two alternate forms.

Corresponding to each type we give the invariant-factors of $|uA + vB|$, show ing that the reduced types depend entirely on the invariant-factors, except in the "singular" case when $|uA + vB| \equiv 0$. The determination of the necessary invariants for the singular case is not considered here, as it has been fully explained by Kronecker and others.*

* Nearly all the papers of Kronecker's which have been quoted give some details as to the proper invariants; the 1890 and 1891 papers are the most complete. The last section of Darboux's paper may also be consulted; and a somewhat different determination of the invariants has been effected by the present writer. (Proc. Lond. Math. Soc., vol. XXXII, 1900, pp. 88 and 326.)

Two symmetric forms.

This case (which is equivalent to that of two quadratic forms) has been handled so often* that what is given here is only included to give completeness to the investigation.

The simple sets of terms are of the types

(i). (G_1): $\qquad A,\ x_1 y_1$ or $x_1 y_2 + x_2 y_1$

corresponding to an invariant-factor u, or two u, u of $|uA + vB|$. There will be no terms in B which contain x_1, y_1, x_2, y_2 in this case. In the second case the form can be reduced to

$$\tfrac{1}{2}(X_1 Y_1 - X_2 Y_2)$$

by writing $\qquad X_1 = x_1 + x_2,\ X_2 = x_1 - x_2.$

(ii). $(G_2 L_1)$ and (L_1):

$$\begin{array}{ll} A, & x_1 y_2 + x_2 y_1 \\ B, & x_2 y_2 \end{array}$$

with an invariant-factor u^2.

Again

$$\begin{array}{ll} A, & x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3, \\ B, & x_2 y_3 + x_3 y_2 \end{array}$$

with two invariant-factors u^2, u^2. But if we write

$$X_1 = x_1 + x_4,\ X_2 = x_2 + x_3,\ X_3 = x_2 - x_3,\ X_4 = x_1 - x_4,$$

the parts in A, B, corresponding to the two invariant-factors, can be separated, thus,*

$$\begin{array}{ll} A, & \tfrac{1}{2}\left[(X_1 Y_2 + X_2 Y_1) + (X_3 Y_4 + X_4 Y_3)\right] \\ B, & \tfrac{1}{2}(X_2 Y_2 - X_3 Y_3). \end{array}$$

(iii). $(G_2 L_2)$ and $(L_2 K_1)$:

$$\begin{array}{ll} A, & x_1 y_2 + x_2 y_1 \\ B, & x_2 y_3 + x_3 y_2 \end{array}$$

giving $\qquad |uA + vB| \equiv 0$

The general forms obtained from the continuation of the process given for $(G_2 L_3)$ and $(L_3 M_1)$ above (p. 245) will be:

First,

$$\begin{array}{ll} A, & x_1 y_2 + x_2 y_1 + \cdots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1} \\ B - cA, & x_2 y_3 + x_3 y_2 + \cdots + x_{2m-2} y_{2m-1} + x_{2m-1} y_{2m-2}. \end{array}$$

* See, in particular, Kronecker, Berliner Monatsberichte, 1874, p. 59 = Ges. Werke, vol. I, p. 351.

corresponding to two invariant-factors $(u+cv)^m$, $(u+cv)^m$ of $|uA+vB|$. The parts corresponding to each factor can be separated by writing

$$X_r = x_r + x_{2m+1-r},\ X_{2m+1-r} = x_r - x_{2m+1-r},$$
$$(r = 1, 2, \ldots, m).$$

Then if m be odd we find a pair of terms in the middle of A of the form $\frac{1}{2}(X_m Y_m - X_{m+1} Y_{m+1})$ and the general form is

$$\begin{aligned} A, &\quad \tfrac{1}{2}(X_1 Y_2 + X_2 Y_1 + X_3 Y_4 + X_4 Y_3 + \ldots + X_m Y_m), \\ &\quad + \tfrac{1}{2}(-X_{m+1} Y_{m+1} + X_{m+2} Y_{m+3} + X_{m+3} Y_{m+2} + \ldots \\ &\qquad\qquad + X_{2m-1} Y_{2m} + X_{2m} Y_{2m-1}), \\ B - cA, &\quad \tfrac{1}{2}(X_2 Y_3 + X_3 Y_2 + \ldots + X_{m-1} Y_m + X_m Y_{m-1}), \\ &\quad + \tfrac{1}{2}(X_{m+1} Y_{m+2} + X_{m+2} Y_{m+1} + \ldots + X_{2m-2} Y_{2m-1} + X_{2m-1} Y_{2m-2}). \end{aligned}$$

But if m be even, we find this pair of terms $\frac{1}{2}(X_m Y_m - X_{m+1} Y_{m+1})$ in the middle of $(B-cA)$ and the forms are,

$$\begin{aligned} A, &\quad \tfrac{1}{2}(X_1 Y_2 + X_2 Y_1 + \ldots + X_{m-1} Y_m + X_m Y_{m-1}), \\ &\quad + \tfrac{1}{2}(X_{m+1} Y_{m+2} + X_{m+2} Y_{m+1} + \ldots + X_{2m-1} Y_{2m} + X_{2m} Y_{2m-1}), \\ B - cA, &\quad \tfrac{1}{2}(X_2 Y_3 + X_3 Y_2 + \ldots + X_m Y_m), \\ &\quad + \tfrac{1}{2}(-X_{m+1} Y_{m+1} + X_{m+2} Y_{m+3} + X_{m+3} Y_{m+2} + \ldots \\ &\qquad\qquad + X_{2m-2} Y_{2m-1} + X_{2m-1} Y_{2m-2}).^* \end{aligned}$$

Second

$$\begin{aligned} A, &\quad x_2 y_3 + x_3 y_2 + \ldots + x_{2m-2} y_{2m-1} + x_{2m-1} y_{2m-2} \\ B, &\quad x_1 y_2 + x_2 y_1 + \ldots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1} \end{aligned}$$

corresponding to the invariant-factors v^m, v^m of $|uA+vB|$. The parts can be separated as in the first case.

Third,

$$\begin{aligned} A, &\quad x_1 y_2 + x_2 y_1 + \ldots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1} + x_{2m+1} y_{2m+1}, \\ B - cA, &\quad x_2 y_3 + x_3 y_2 + \ldots + x_{2m} y_{2m+1} + x_{2m+1} y_{2m} \end{aligned}$$

corresponding to the single invariant-factor $(u+cv)^{2m+1}$ of $|uA+vB|$.

Fourth,

$$\begin{aligned} A, &\quad x_1 y_2 + x_2 y_1 + \ldots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1}, \\ B - cA, &\quad x_2 y_3 + x_3 y_2 + \ldots + x_{2m} y_{2m}, \end{aligned}$$

corresponding to the single invariant-factor $(u+cv)^{2m}$.

The *fifth* and *sixth* cases correspond to the invariant-factors v^{2m+2}, v^{2m} and are obtained from the third and fourth by putting $c=0$ and interchanging A, B.

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*Kronecker states in the paper just quoted (Ges. W. I. p. 367) that this division into two parts is only possible if $m=2n$ or is *even*; but apparently this is an oversight, as on p. 354 (c) he makes no restriction on m.

Seventh,
$$A, \quad x_1 y_2 + x_2 y_1 + \dots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1},$$
$$B, \quad x_2 y_3 + x_3 y_2 + \dots + x_{2m} y_{2m+1} + x_{2m+1} y_{2m}$$

corresponding to $|uA + vB| \equiv 0$. Here we have obtained no rule for finding m directly from the determinant without carrying out the reduction (see footnote to the next class of reductions).

A Symmetric and an Alternate Form.

The simple sets of terms first found will give types as below:

(i). (G_1): (terms belonging to the symmetric form only),

$$A, \quad x_1 y_1 \text{ or } x_1 y_2 + x_2 y_1.$$

In the first case, we have an invariant-factor u of $|uA + vB|$; in the second, two u, u. The second case is equivalent to two of the first, by writing it in the form

$$\tfrac{1}{2}\left[(x_1 + x_2)(y_1 + y_2) - (x_1 - x_2)(y_1 - y_2)\right].$$

Terms belonging to the alternate form only:

$$B, \quad x_1 y_2 - x_2 y_1.$$

Here we have two invariant-factors v, v, and the form *cannot* be split up into two, one for each invariant-factor.

(ii). $(G_2 L_1)$ and (L_1):

$$A, \quad x_1 y_2 + x_2 y_1 + x_3 y_4 + x_4 y_3,$$
$$B, \quad x_2 y_3 - x_3 y_2.$$

Here we have two invariant-factors u^2, u^2 which cannot be separated Again we may have, interchanging the parts played by A and B,

$$A, \quad x_2 y_3 + x_3 y_2,$$
$$B, \quad x_1 y_2 - x_2 y_1 + x_3 y_4 - x_4 y_3,$$

with two invariant-factors v^2, v^2; these can be separated by writing

$$x_1 + x_4 = X_1, \quad x_1 - x_4 = X_4, \quad x_2 - x_3 = X_2, \quad x_2 + x_3 = X_3,$$

with the corresponding substitutions for the y's. Then the types become

$$A, \quad \tfrac{1}{2}\left[-X_2 Y_2 + X_3 Y_3\right],$$
$$B, \quad \tfrac{1}{2}\left[X_1 Y_2 - X_2 Y_1 + Y_3 X_4 - Y_4 X_3\right],$$

in which the parts are separable. Finally, we may have

$$\begin{aligned} &A, \quad x_2 y_2, \\ &B, \quad x_1 y_2 - x_2 y_1, \end{aligned}$$

with an invariant-factor v^2.

(iii). $(G_2 L_2)$ and $(L_2 K_1)$:

$$\begin{aligned} &A, \quad x_1 y_2 + x_2 y_1, \\ &B, \quad x_2 y_3 - x_3 y_2, \end{aligned}$$

and here $|uA + vB| \equiv 0$ for all values of u, v.

This exhausts all the possibilities for the specially simple types; we proceed to examine the more general forms, of which the foregoing are particular cases. By continuing the process indicated above for dealing with $(G_2 L_3)$ and $(L_3 M_1)$, we obtain the following possible cases:

First:

$$\begin{aligned} &A, \quad x_1 y_2 + x_2 y_1 + \ldots\ldots + x_{2m-1} y_{2m} + x_{2m} y_{2m-1}, \\ &B, \quad x_2 y_3 - x_3 y_2 + \ldots\ldots + x_{2m-2} y_{2m-1} - x_{2m-1} y_{2m-2}, \end{aligned}$$

a case which corresponds to two invariant-factors u^m, u^m of $|uA + vB|$; if m be odd, the parts corresponding to the two may be separated, but not if m be even* (the reason being that there are $(m-1)$ pairs of terms in B, and when m is odd, these can be divided into two sets each containing $\frac{1}{2}(m-1)$ pairs).

Second:

$$\begin{aligned} &A, \quad x_2 y_3 + x_3 y_2 + \ldots\ldots + x_{2m-2} y_{2m-1} + x_{2m-1} y_{2m-2}, \\ &B, \quad x_1 y_2 - x_2 y_1 + \ldots\ldots + x_{2m-1} y_{2m} - x_{2m} y_{2m-1}, \end{aligned}$$

corresponding to the factors v^m, v^m of $|uA + vB|$; if m be even, the two parts can be separated, but not if m be odd (for here there are m pairs in B). The separating substitutions are analogous to those required for the last case.

Third:

$$\begin{aligned} &(A), \quad (x_1 y_2 + x_2 y_1) + \ldots\ldots + (x_{2m-2} y_{2m} + x_{2m} y_{2m-1}) + x_{2m+1} y_{2m+1}, \\ &(B), \quad (x_2 y_2 - x_3 y_2) + \ldots\ldots + (x_{2m} y_{2m+1} - x_{2m+1} y_{2m}), \end{aligned}$$

corresponding to the single invariant-factor u^{2m+1} of $|uA + vB|$.

Fourth:

$$\begin{aligned} &(A), \quad (x_2 y_3 + x_3 y_2) + \ldots\ldots + (x_{2m-2} y_{2m-1} + x_{2m-1} y_{2m-2}) + x_{2m} y_{2m}, \\ &(B), \quad (x_1 y_2 - x_2 y_1) + \ldots\ldots + (x_{2m-1} y_{2m} - x_{2m} y_{2m-1}), \end{aligned}$$

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* If we write

$$\left.\begin{aligned} \xi_k &= x_k - (-1)^{m-k} x_{2m+1-k} \\ \xi_{m+k} &= x_k + (-1)^{m-k} x_{2m+1-k} \end{aligned}\right\}, \quad (k = 1, 2, \ldots\ldots, m).$$

and make the congruent substitutions for the η's in term of the y's, we find that A, B divide into the parts

$$\begin{aligned} &A, \quad [\tfrac{1}{2}(\xi_1 \eta_2 + \xi_2 \eta_1) + \ldots\ldots + \tfrac{1}{2} \xi_m \eta_m] + [\tfrac{1}{2}(\xi_{m+1} \eta_{m+2} + \xi_{m+2} \eta_{m+1}) + \ldots\ldots - \tfrac{1}{2} \xi_{2m} \eta_{2m}], \\ &B, \quad [\tfrac{1}{2}(\xi_2 \eta_3 - \xi_3 \eta_2) + \ldots\ldots + \tfrac{1}{2}(\xi_{m-1} \eta_m - \xi_m \eta_{m-1})] + [\tfrac{1}{2}(\xi_{m+2} \eta_{m+3} - \xi_{m+3} \eta_{m+2}) + \ldots + \tfrac{1}{2}(\xi_{2m-1} \eta_{2m} - \xi_{2m} \eta_{2m-1})]. \end{aligned}$$

for the single invariant-factor v^{2m} of $|uA+vB|$.

$$\textit{Fifth:}\quad \begin{array}{ll} (A), & (x_1y_2+x_2y_1)+\ldots\ldots+(x_{2m-1}y_{2m}+x_{2m}y_{2m-1}), \\ (B), & (x_2y_3-x_3y_2)+\ldots\ldots+(x_{2m}y_{2m+1}-x_{2m+1}y_{2m}); \end{array}$$

here $|uA+vB|\equiv 0$*.

We have now exhausted all the types that can be found by the first method of investigation indicated above; the complete reduction, corresponding to non-zero roots of $|\lambda A-B|=0$, has to be effected in a different way, as already explained. Thus we find the type of reduced terms:

$$\textit{Sixth:}\quad \begin{array}{ll} (A), & (x_1y_2+x_2y_1)+\ldots\ldots+(x_{2s-1}y_{2s}+x_{2s}y_{2s-1}), \\ (B), & c(x_1y_2-x_2y_1)+\ldots\ldots+c(x_{2s-1}y_{2s}-x_{2s}y_{2s-1}) \\ & +(x_2y_3-x_3y_2)+\ldots\ldots+(x_{2s-2}y_{2s-1}-x_{2s-1}y_{2s-2}), \end{array}$$

corresponding to the pair of invariant factors $(\lambda-c)^s$, $(\lambda+c)^s$ of $|\lambda A-B|$.

Two Alternate Forms.

The simple types are here:

(i). (G): $\qquad A,\quad x_1y_2-x_2y_1,$

with two invariant-factors u, u of $|uA+vB|$.

(ii). (G_2L_1) and (L_1):

$$\begin{array}{ll} A, & x_1y_2-x_2y_1+x_3y_4-x_4y_3 \\ B, & x_2y_3-x_3y_2, \end{array}$$

giving two invariant-factors u^2, u^2.

(iii). (G_2L_2) and (L_2K_1):

$$\begin{array}{ll} A, & x_1y_2-x_2y_1, \\ B, & x_2y_3-x_3y_2, \end{array}$$

corresponding to the simplest types that give $|uA+vB|\equiv 0$.

*It is one drawback to the method explained here, that the number m cannot be determined (so far as we have shown) from the determinant $|uA+vB|$ and its minors. According to the foregoing, it would seem to be necessary to calculate the reduced form before we can find m, but this is not the case, as will be seen from Kronecker's paper and those quoted above. I shall not give the rules for its determination, which will be found in Muth's "Elementartheiler" (p. 108), where m is called a "Minimalgradzahl." Note that, in all the problems examined in this paper, the two series of m's given by Muth are necessarily the same.

We have only to add the results obtained by continuing the process already indicated for dealing with $(G_2 L_3)$ and $(L_3 M_1)$; we obtain a smaller number of types than in the case of a symmetric and an alternate form.
They are, in order:

First, $\quad A,\quad x_1 y_2 - x_2 y_1 + \dots + x_{2m-1} y_{2m} - x_{2m} y_{2m-1},$

$\quad B - cA,\quad x_2 y_3 - x_3 y_2 + \dots + x_{2m-2} y_{2m-1} - x_{2m-1} y_{2m-2},$

corresponding to the two invariant-factors $(u + cv)^m$, $(u + cv)^m$, of $|uA + vB|$. (Here c may be zero.)

Second, $\quad A,\quad x_2 y_3 - x_3 y_2 + \dots + x_{2m-2} y_{2m-1} - x_{2m-1} y_{2m-2},$

$\quad B,\quad x_1 y_2 - x_2 y_1 + \dots + x_{2m-1} y_{2m} - x_{2m} y_{2m-1},$

corresponding to the two invariant-factors v^m, v^m, of $|uA + vB|$.

Third, $\quad A,\quad x_1 y_2 - x_2 y_1 + \dots + x_{2m-1} y_{2m} - x_{2m} y_{2m-1},$

$\quad B,\quad x_2 y_3 - x_3 y_2 + \dots + x_{2m} y_{2m+1} - x_{2m+1} y_{2m},$

corresponding to $|uA + vB| \equiv 0$. (For the meaning of m see the last footnote.)

It will be observed that here the invariant-factors *always* occur in pairs; and not, as in the case of a symmetric and an alternate form, sometimes singly.

As an illustration of the methods explained we may take the two forms,

$$A = a(x_2 y_3 - x_3 y_2) + b(x_3 y_1 - x_1 y_3) + c(x_1 y_2 - x_2 y_1)$$
$$+ p(x_1 y_4 - x_4 y_1) + q(x_2 y_4 - x_4 y_2) + r(x_3 y_4 - x_4 y_3).$$

$B = x_1 y_1 + x_2 y_2 + x_3 y_3.$

Let us write
$$-\xi_4 = \frac{\partial A}{\partial y_3} = -rx_4 - bx_1 + ax_2,$$
$$\xi_3 = \frac{\partial A}{\partial y_4} = px_1 + qx_2 + rx_3,$$

then, as in §1, we have
$$A = \xi_3\eta_4 - \xi_4\eta_3 + \frac{\theta}{r}(x_1 y_2 - x_2 y_1),$$
where
$$\theta = ap + bq + cr,$$

We may note that θ is the Pfaffian of $|A|$. Substituting in B for x_3, y_3 in terms of ξ_3, η_3 we have
$$B = x_1 y_1 + x_2 y_2 + \frac{1}{r^2}[\xi_3 - (px_1 + qx_2)][\eta_3 - (py_1 + qy_2)];$$

if we apply §2 to B we find that with,
$$\xi_1 = x_1 - \frac{p}{\sigma^2}\xi_3,\ \xi_2 = x_2 - \frac{q}{\sigma^2}\xi_3,$$
$$\sigma^2 = p^2 + q^2 + r^2,$$

we have

$$B = \xi_1 \eta_1 + \xi_2 \eta_2 + \frac{1}{r^2}(p\xi_1 + q\xi_2)(p\eta_1 + q\eta_2) + \frac{1}{\sigma^2}\xi_3 \eta_3$$

Substitute in A for x_1, y_1, x_2, y_2, in terms of ξ_1, η_1, ξ_2, η_2 and then

$$A = \xi_3 Y_4 - \eta_3 X_4 + \frac{\theta}{r}(\xi_1 \eta_2 - \xi_2 \eta_1),$$

where $$X_4 = \xi_4 + \frac{\theta}{r\sigma^2}(p\xi_2 - q\xi_1),$$

Turning again to B, we have, if

$$\bar{\xi}_1 = \xi_1 + \frac{pq}{p^2 + r^2}\xi_2$$

$$B = \frac{p^2 + r^2}{r^2}\bar{\xi}_1 \bar{\eta}_1 + \frac{\sigma^2}{p^2 + r^2}\xi_2 \eta_2 + \frac{1}{\sigma^2}\xi_3 \eta_3$$

and this substitution does not alter the form of A.

Finally write,

$$X_1 = (p^2 + r^2)^{\frac{1}{2}}\bar{\xi}_1 / r,\ X_2 = \sigma\xi_2 / (p^2 + r^2)^{\frac{1}{2}},\ X_3 = \xi_3 / \sigma,$$

and then we have

$$A = \sigma(X_3 Y_4 - X_4 Y_3) + \frac{\theta}{\sigma}(X_1 Y_2 - X_2 Y_1),$$

$$B = X_1 Y_1 + X_2 Y_2 + X_3 Y_3,$$

It will be observed that this form of reduction is by no means unique. For we may obviously take instead of X_4 the linear function $(X_4 + \alpha X_3)$, where α is arbitrary; and in place of X_1 and X_2, $X_1 \cos\beta - X_2 \sin\beta$, and $X_1 \sin\beta + X_2 \cos\beta$, where β is also arbitrary but real. We can make the function $(X_4 + \alpha X_3)$ symmetrical in x_1, x_2, x_3 by writing $\alpha = (aq - bp)/r\sigma^2$.

This agrees with what we know from the geometrical interpretation; for the problem is the reduction of a linear complex to its central axis.

5.—*Hermite's Forms.*

These are bilinear forms such as $\Sigma a_{rs} x_r y_s$, in which a_{rs}, a_{sr} are conjugate imaginaries (in particular a_{rr} is real), and x_r, y_r are also conjugate imaginaries. The methods explained above can be applied to the reduction of a pair of Hermite's forms, A, B, in which the substitutions on the x's and y's are conjugate imaginaries.

For consider the methods of §1; if $a_{11} \neq 0$ we have, as there shewn

$$A = \frac{1}{a_{11}} \frac{\partial A}{\partial x_1} \frac{\partial A}{\partial y_1} + \frac{1}{a_{11}^2} B,$$

where B is a Hermite's form in $2(n-1)$ variables. But if we write

$$X_1 = \frac{\partial A}{\partial y_1} = a_{11}x_1 + a_{21}x_2 + \ldots\ldots + a_{n1}x_n,$$
$$Y_1 = \frac{\partial A}{\partial x_1} = a_{11}y_1 + a_{12}y_2 + \ldots\ldots + a_{1n}y_n,$$

we see by the definitions of the coefficients that these two substitutions are conjugate imaginaries, and then

$$A = \frac{1}{a_{11}} X_1 Y_1 + \frac{1}{a_{11}^2} B.$$

Again if $a_{11} = 0$, we find, as in §1, that

$$A = \frac{1}{a_{12}} \frac{\partial A}{\partial x_1} \frac{\partial A}{\partial y_1} + \frac{1}{a_{21}} \frac{\partial A}{\partial x_2} \frac{\partial A}{\partial y_1} - \frac{a_{22}}{a_{12}a_{21}} \frac{\partial A}{\partial x_1} \frac{\partial A}{\partial y_1} - \frac{1}{a_{12}a_{21}} B,$$

where B is a Hermite's form in $(n-2)$ variables. But, since a_{12}, a_{21} are conjugate imaginaries their product is real and positive; thus if we write

$$X_1 = \frac{1}{a_{12}} \frac{\partial A}{\partial y_2} - \frac{1}{2} \frac{a_{22}}{a_{12}a_{21}} \frac{\partial A}{\partial y_1}, \quad X_2 = \frac{\partial A}{\partial y_1},$$
$$Y_1 = \frac{1}{a_{21}} \frac{\partial A}{\partial x_2} - \frac{1}{2} \frac{a_{22}}{a_{12}a_{21}} \frac{\partial A}{\partial x_1}, \quad Y_2 = \frac{\partial A}{\partial x_1},$$

the substitutions are conjugate imaginaries, for a_{22} is real. Then

$$A = X_1 Y_2 + X_2 Y_1 - \frac{1}{a_{12}a_{21}} B.$$

Passing to the methods of §2, we must first see that the minor D_{11} is real; to prove this we note that the change of $+i$ to $-i$ in D_{11} will only change rows into columns and so will not alter D_{11}. By a similar argument the minors D_{12} D_{21} are conjugate imaginaries.

In the first place, if $D_{11} \neq 0$ we have

$$A = \frac{D}{D_{11}} x_1 y_1 + C,$$

where C is derived from A by writing zero for x_1, y_1 and X_r, Y_r for x_r, y_r $(r=2, 3, \ldots, n)$. Further X_r, Y_r are defined by

$$\frac{\partial C}{\partial X_r} = \frac{\partial A}{\partial x_r}, \frac{\partial C}{\partial Y_r} = \frac{\partial A}{\partial y_r} \quad (r=2, 3, \ldots, n),$$

from which it readily follows that the substitutions are conjugate imaginaries.

Again, if $D_{11}=0$, $(D \neq 0)$, we find

$$A = \frac{D}{D_{12}} x_1 y_2 + \frac{D}{D_{21}} x_2 y_1 - \frac{D D_{22}}{D_{12} D_{21}} x_1 y_1 + C,$$

where C is found by writing in A zero for x_1, y_1, x_2, y_2 and X_r, Y_r for x_r, y_r, $(r=3, 4, \ldots, n)$.

If now

$$X_2 = \frac{D}{D_{21}} x_2 - \tfrac{1}{2} \frac{D D_{22}}{D_{12} D_{21}} x_1,$$

$$Y_2 = \frac{D}{D_{12}} y_2 - \tfrac{1}{2} \frac{D D_{22}}{D_{12} D_{21}} y_1,$$

we have

$$A = x_1 Y_2 + x_2 Y_1 + C,$$

and X_2, Y_2 are conjugate imaginaries; as before, X_r, Y_r are conjugate imaginaries.

Thus, if $|A|=0$ or $|B|=0$, we can reduce a pair of Hermite's forms (A, B) by a process analogous to that given before for symmetric forms (or quadratics); also, if $\lambda = c$ be a *real* root of $|\lambda A - B| = 0$, the same method can be applied, for $(cA - B)$ is then a Hermite's form. But, in general, some of the roots of $|\lambda A - B|$ will not be real; and if c be complex, $(cA - B)$ is no longer a Hermite's form. Thus, our general process of reduction fails for these complex roots; and, to carry out the reduction, we must proceed as in the case of a sym metric and an alternate form above. There are certain obvious changes, but the reductions can be arranged to correspond step by step; the final typical reduced sets of terms being

$$\begin{aligned}
&(A), \quad (x_1 y_2 + x_2 y_1) + \cdots + (x_{2s-1} y_{2s} + x_{2s} y_{2s-1}), \\
&(B), \quad (c x_1 y_2 + c_0 x_2 y_1) + \cdots + (c x_{2s-1} y_{2s} + c_0 x_{2s} y_{2s-1}) \\
&\qquad + (x_2 y_3 + x_3 y_2) + \cdots + (x_{2s-2} y_{2s-1} + x_{2s-1} y_{2s-2}),
\end{aligned}$$

corresponding to the pair of invariant-factors $(\lambda - c)^s$, $(\lambda - c_0)^s$, c_0 being the conjugate imaginary to c. The other types are precisely the same as those for symmetric forms and will not be repeated now.

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